

Globularily generated double categories

Juan Orendain

Abstract: We study globularily generated double categories. The condition of a double category being globularily generated is a finiteness condition generalizing the condition of a double category being trivial. We establish analogies between the way trivial double categories and globularily generated double categories relate to general double categories. We organize globularily generated double categories into a 2-category and we prove, among other things, that 2-category of globularily generated double categories is a strictly 2-reflective sub 2-category of an appropriate sub 2-category of 2-category of double categories.

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1 Introduction

There exists, in the mathematical literature, a variety of competing ideas of what a higher order categorical structure should be. This set of ideas reduces, in the case of categorical structures of order 2, to the concepts of bicategory and double category, both types of structures introduced by Ehresmann, in [3] and [4]. These two concepts are related in different ways. The most obvious being that every bicategory can be considered as a 'trivial' double category. Less obviously, every double category admits an adjacent trivial double category, its horizontal bicategory. These relations admit first order categorical extensions. These relations can be summed up by the following statement: Category adjacent to 3-category of bicategories, bifunctors, lax natural transformations, and deformations is a reflective subcategory of category adjacent to 2-category of double categories, double functors, and

double natural transformations, with horizontalization functor as reflector. Second order extensions to this statement are studied in [7].

We study globularly generated double categories. We regard the condition of a double category being globularly generated as the condition of double category being 'almost trivial.' Our aim is to formally articulate this idea and to establish analogies between the way trivial double categories and globularly generated double categories relate to the general concept of double category. More precisely, we organize globularly generated double categories into a sub 2-category of 2-category of double categories, and we prove this sub 2-category is 2-reflective. We explicitly construct a 2-reflector which we regard as a globularly generated analog of horizontalization functor defined in the trivial case [7]. We call this 2-functor the globularly generated piece 2-functor. We compute the globularly generated piece of classic examples of double categories. We now sketch the contents of this paper.

In section 2 we recall some of the basic concepts related to the theory of double categories. We present relevant examples and set notation used in the rest of the paper. In section 3 we define and study the concept of globularly generated double category. We establish the technical framework needed for results in subsequent sections. We construct the vertical filtration of category of morphisms of a globularly generated double category and we define the notion of vertical length of a globularly generated 2-morphism. With the aid of these concepts we establish technical results describing relations between different types of 2-morphisms in globularly generated double categories. In section 4 we define and study the concept of globularly generated piece of a double category. We extend this construction to a 2-categorical setting. We define the globularly generated piece 2-functor and the diagram of vertical functors. We generalize the fact that sequence of vertical categories of a double category defines a filtration of its category of morphisms to a categorical setting by proving that morphism category of globularly generated piece 2-functor is the limit of sequence of vertical functors. Moreover, we prove that globularly generated piece 2-functor is a strict 2-reflector, thus generalizing the relation between bicategories and double categories established by horizontalization functor. Finally, in sections 5 we perform computations of the globularly generated piece of double categories presented in section 2. Precisely, we compute globularly generated piece of double category of algebras and bimodules and we compute globularly generated piece of double category of closed manifolds and cobordisms.

The exposition of the paper will be elementary. We assume nevertheless that the reader is familiar with the basic notions of the theory of bicategories,

bifunctors, and transformations. We refer the reader to [5] for basic notation and definitions. We follow notational conventions in the theory of double categories for the most part. We refer the reader to [7] for a survey on notational conventions in the theory of double categories.

2 Double categories

In this first section we establish the theoretical and notational framework needed for the rest of the paper. We recall the concepts of double category, double functor, and double natural transformation. We present relevant examples and set notational conventions for subsequent sections.

Definition 2.1. We understand, for a double category, a category weakly internal to 2-category **Cat** of categories, functors, and natural transformations. More precisely, a double category C consists of the following data:

1. **Objects and morphisms:** Categories C_0, C_1 . We call C_0 the category of objects of C and C_1 the category of morphism of C .
2. **Source and target:** Functors $s, t : C_1 \rightarrow C_0$. We call s and t the source and target functors of C respectively.
3. **Horizontal identity:** Functor $i : C_0 \rightarrow C_1$. We call i the horizontal identity functor of C .
4. **Horizontal composition:** Bifunctor $* : C_1 \times_{C_0} C_1 \rightarrow C_1$, where fibration in $C_1 \times_{C_0} C_1$ is taken with respect to pair s, t . We call $*$ the horizontal composition bifunctor of C .
5. **Identity transformations:** Natural isomorphisms $\lambda : *(is \times id_{C_1}) \rightarrow id_{C_1}$ and $\rho : *(id_{C_1} \times it) \rightarrow id_{C_1}$. We call λ and ρ the left and right identity transformations of C respectively.
6. **Associator:** Natural isomorphism $\xi : *(* \times id_{C_1}) \rightarrow *(id_{C_1} \times *)$. We call ξ the associator of C .

We require left and right identity transformations and associator of C to be related by McLane's triangular and pentagonal axioms [6]. Moreover, we require that source and target of each component of left and right identity transformations and associator of C be identity endomorphisms in C_0 .

We call objects and morphisms of category of objects C_0 of a double category C objects and vertical morphisms of C respectively. We call objects and morphisms of category of morphisms of double category C horizontal morphisms and 2-morphisms of C respectively. We write \circ for composition in category of morphisms C_1 of C and we call \circ the vertical composition operation in C . We call the image, under source and target functors s and t of C , of any 2-morphism in C , its source and its target respectively. We will call the image, under horizontal identity functor i of C , of any object or any vertical morphism of C , its horizontal identity. Finally, we will call the image, under horizontal composition bifunctor $*$ of C , of a compatible pair of horizontal or 2-morphisms the horizontal composition of the pair. The existence of left and right identity transformations for C can be interpreted by saying that horizontal identity functor of C acts as a left and right identity for horizontal composition up to natural isomorphisms. The existence of associator of C can be interpreted by saying that horizontal composition in C is associative up to a natural isomorphism. We say that a 2-morphism in double category C is **globular** if its source and target are vertical identity endomorphisms. The last condition in the definition of a double category can be interpreted by saying that components of left and right identity transformations and of associator of a double category are globular. In the case in which horizontal identity transformations and associator of double category C are identity natural transformations we say that double category C is strict. The following are the main examples of double categories that will be used throughout the paper.

Bicategories: Let B be a bicategory. Denote by \overline{B} pair formed by discrete category generated by collection of 0-cells B_0 of B and category whose collection of objects and whose collection of morphisms are collection of 1-cells of B and collection of 2-cells of B respectively. Denote by i functor generated by function associating, to every 0-cell in B its identity 1-cell in B and denote by $*$ bifunctor generated by horizontal composition of 1- and 2-cells in B . With this structure \overline{B} is a double category. Identity transformations and associator in \overline{B} are defined in terms of those defining the structure of bicategory in B . Double category \overline{B} is strict if and only if bicategory B is a 2-category. We call double categories arising from bicategories in this way trivial double categories. Observe that every 2-morphism in a trivial double category is globular. Every double category such that all its 2-morphisms are globular is trivial.

Algebras: Let \mathbf{Alg}_0 denote category whose collection of objects is collection

of complex algebras and whose collection of morphisms is collection of unital algebra morphisms. Given algebras A, B, C and D , left-right A - B bimodule M and left-right C - D -bimodule N , we say that a triple (f, Φ, g) , where f is a unital algebra morphism from A to C , g is a unital algebra morphism from B to D and Φ is a linear transformation from M to N ; is an equivariant bimodule morphism from M to N , if for every $a \in A$, $b \in B$, and $x \in M$ equation $\Phi(axb) = f(a)\Phi(x)g(b)$ holds. Composition of equivariant bimodule morphisms is performed entry-wise. Let \mathbf{Alg}_1 denote category whose collection of objects is collection of bimodules over complex algebras and whose collection of morphisms is collection of equivariant bimodule morphisms. Denote by \mathbf{Alg} pair formed by categories \mathbf{Alg}_0 and \mathbf{Alg}_1 . Denote by i functor from \mathbf{Alg}_0 to \mathbf{Alg}_1 , associating algebra A as a left-right A -bimodule to every algebra A , and associating equivariant bimodule morphism (f, f, f) to every unital algebra morphism f . Given algebras A, B and C , left A - B bimodule M and left-right B - C bimodule N write $N * M$ for tensor product $M \otimes_B N$ relative to B . Given equivariant bimodule morphisms (f, Φ, g) and (g, Ψ, h) we write $(g, \Psi, h) * (f, \Phi, g)$ for tensor product $(f, \Phi \otimes_g \Psi, h)$. Denote by $*$ bifunctor, from $\mathbf{Alg}_1 \times_{\mathbf{Alg}_0} \mathbf{Alg}_1$ to \mathbf{Alg}_1 defined by operations $*$ defined above. This structure provides pair \mathbf{Alg} of categories \mathbf{Alg}_0 and \mathbf{Alg}_1 with the structure of a double category. Left and right identity transformations and associator in \mathbf{Alg} are defined by those of relative tensor product bifunctor. An analogous structure is defined in [1] where algebras are replaced by von Neumann algebras with finite dimensional center. Denote by \mathbf{vN}_0^f category whose objects are von Neumann algebras with finite dimensional center and whose morphisms are finite index morphisms. Denote by \mathbf{vN}_1^f category whose objects are bimodules over von Neumann algebras with finite dimensional center and whose morphisms are bounded intertwiners. Denote by \mathbf{vN}^f pair formed by \mathbf{vN}_0^f and \mathbf{vN}_1^f . Denote by i and $*$ the Haagerup standard form functor and the Connes fusion operation bifunctor. Both functors defined in [1]. In analogy with the case of bicategory \mathbf{Alg} this pair of functors provides pair \mathbf{vN}^f with the structure of a bicategory.

Cobordisms: Let n be a positive integer. Let $\mathbf{Cob}(n)_0$ denote category whose collection of objects is collection of closed n -dimensional smooth manifolds and whose collection of morphisms is collection of diffeomorphisms between manifolds. Given closed n -manifolds X, Y, Z and W , a cobordism M from X to Y , and a cobordism N from Z to W , we will say that a triple (f, Φ, g) , where f is a diffeomorphism from X to Z , g is a diffeomorphism from Y to W , and where Φ is a diffeomorphism from M to N ; is an equiv-

ariant diffeomorphism from M to N if restriction of Φ to X equals f and restriction of Φ to Y equals g . Composition of equivariant morphisms between cobordisms is performed entry-wise. Let $\mathbf{Cob}(n)_1$ denote category whose collection of objects is collection of cobordisms between closed n -dimensional manifolds and whose collection of morphisms is collection of equivariant diffeomorphisms between cobordisms. Given an n -dimensional manifold X we write i_X for cobordism $X \times [0, 1]$. Given a diffeomorphism $f : X \rightarrow Y$ between closed n -dimensional manifolds X and Y , we denote by i_f equivariant diffeomorphism $(f, f \times [0, 1], f)$ from cobordism i_X to cobordism i_Y . These two functions define a functor from $\mathbf{Cob}(n)_0$ to $\mathbf{Cob}(n)_1$. Denote this functor by i . Given compatible cobordisms M and N with respect to manifold X we write $N * M$ for joint union $M \cup_X N$ with respect to X . Finally, given equivariant diffeomorphisms (f, Φ, g) and (g, Ψ, h) , compatible with respect to diffeomorphism g denote by $(g, \Psi, h) * (f, \Phi, g)$ joint union $(f, \Phi \cup_g \Psi, h)$ of (f, Φ, g) and (g, Ψ, h) with respect to g . The two operations defined above form a bifunctor which we denote by $*$. With this structure pair $\mathbf{Cob}(n)$ is a double category, where identity transformations and associator come from the obvious diffeomorphisms from cobordism theory.

We now describe how to organize double categories into a 2-category. We begin with the following definition.

Definition 2.2. Let C, D be double categories. We understand for a double functor from C to D , a functor from C to D , internal to 2-category \mathbf{Cat} of categories, functors, and natural transformations. More precisely, a double functor $F : C \rightarrow D$ from C to D consists of the following data:

1. **Components:** Functors $F_0 : C_0 \rightarrow D_0$ and $F_1 : C_1 \rightarrow D_1$. We call F_0 the object functor of F and we call F_1 the morphism functor of F .
2. **Unit transformation:** Natural isomorphism $\mu^F : F_1 i \rightarrow i F_0$. We call μ the unit transformation of F .
3. **Composition:** Natural isomorphism $\nu^G : *(F_1 \times F_1) \rightarrow F_1 *$. We call ν the horizontal composition transformation of F .

We require object and morphism functors of F to intertwine source and target functors of C and D . We require components of unit and horizontal composition transformations of F to be globular isomorphisms. Finally, we require unit and horizontal composition transformations of F to satisfy McLane's coherence conditions for a monoidal functor [6].

Composition of double functors is defined component-wise. This composition operation is associative and unital. When unit and horizontal composition transformations of a double functor F are identity natural isomorphisms we say that F is a strict double functor. We now describe cells between double functors.

Definition 2.3. Let C, D be double categories. Let $F, G : C \rightarrow D$ be double functors from C to D . We will understand for a double natural transformation from F to G a natural transformation from F to G internal to 2-category **Cat** of categories, functors, and natural transformations. More precisely, a double natural transformation $\eta : F \rightarrow G$ from F to G consists of a pair of natural transformations $\eta_0 : F_0 \rightarrow G_0$ and $\eta_1 : F_1 \rightarrow G_1$. We call η_0 the object natural transformation of η and η_1 the morphism natural transformation of η . We require η_0 and η_1 to satisfy the following conditions:

1. **Source and target:** Let f be a horizontal morphism of C with domain and codomain a and b respectively. In that case the following equations hold.

$$s\eta_f = \eta_x \text{ and } t\eta_f = \eta_y$$

2. **Horizontal identity:** Let x be an object in C . In that case the following equation holds.

$$\eta_{i_x} \mu^F = \mu^G i_{\eta_x}$$

3. **Horizontal composition:** Let f, g be a composable pair of horizontal morphisms of C . In that case the following equation holds.

$$\eta_{g*f} \nu_{g*f}^F = \nu_{Gg*Gf}^G (\eta_g * \eta_f)$$

Vertical and horizontal compositions of compatible double natural transformations is performed component-wise. With these operations collection of double categories, collection of double natural transformations, and collection of double natural transformations form collection of 0-, 1-, and 2-cells of a 2-category respectively. We denote this 2-category by **dCat**.

3 Globularily generated double categories

In this section we introduce the condition of a double category being globularily generated. We develop the technical tools necessary to obtain results on the structure of globularily generated double categories.

We will say that a pair D of categories D_0, D_1 is a sub-double category of a double category C if D_0 is a subcategory of object category C_0 of C , D_1 is a subcategory of morphism category C_1 of C , source and target functors s, t of C restrict to functors from D_1 to D_0 , horizontal identity functor i of C restricts to a functor from D_0 to D_1 , horizontal composition bifunctor $*$ restricts to a bifunctor from $D_1 \times_{D_0} D_1$ to D_1 , components of left and right identity transformations of C , associated to horizontal morphisms in D_1 , are morphisms in D_1 , and if components of associator of C , associated to composable triples of horizontal morphisms of C in D_1 , are morphisms in D_1 . Observe that in this case pair D together with restrictions of structure functors s, t, i , and $*$ of C , left and right identity natural isomorphisms, and associator of C , is itself a double category. We say that a sub-double category D of a double category C is complete, if collections of objects, vertical morphisms, and horizontal morphisms of D are equal to collections of objects, vertical morphisms, and horizontal morphisms of C respectively.

Given a collection of sub-double categories D_α , $\alpha \in A$, of a double category C , pair formed by intersection $\cap_{\alpha \in A} D_{\alpha_0}$ of object categories of double categories D_α , $\alpha \in A$, and intersection $\cap_{\alpha \in A} D_{\alpha_1}$ of morphism categories of double categories D_α , $\alpha \in A$, is again a sub-double category of C . Given a collection of 2-morphisms X of double category C , we call the intersection of all complete sub-double categories D , of C , such that collection X is contained in collection of 2-morphisms of D , the complete sub-double category of C generated by X . We say that a collection of 2-morphisms X of double category C generates C if C is equal to complete sub-double category of C generated by X . The following is the main definition of this section.

Definition 3.1. Let C be a double category. We say that C is a globularily generated double category if C is generated by its collection of globular 2-morphisms.

We now proceed to the introduction of our main technical tool in proving results concerning globularily generated double categories, namely, the vertical length of 2-morphisms. We begin by recursively associating, to every globularily generated double category, a filtration of its category of morphisms as follows:

Let C be a globularly generated double category. Denote by H_1^C the union of collection of globular 2-morphisms of C and collection of horizontal identities of vertical morphisms of C . Write V_1^C for subcategory of category of morphisms C_1 of C , generated by collection H_1^C , that is, V_1^C denotes subcategory of C_1 whose morphisms are vertical compositions of globular 2-morphisms and horizontal identities of C . Let n be an integer strictly greater than 1. Suppose category V_{n-1}^C has been defined. We now define category V_n^C . First denote by H_n^C collection of all possible horizontal compositions of 2-morphisms in category V_{n-1}^C . We make, in that case, category V_n^C to be subcategory of category of morphisms C_1 of C , generated by collection H_n^C . That is, category V_n^C is subcategory of C_1 whose collection of morphisms is collection of vertical compositions of elements of collection H_n^C .

We have thus associated, to every double category C , a sequence of subcategories $\{V_n^C\}$ of category of morphisms C_1 of C . We call, for every n , category V_n^C the n -th vertical category associated to double category C . We have used, in the above construction, for every n , an auxiliary collection of 2-morphisms H_n^C of C . Observe that for each n collection H_n^C both contains the horizontal identity of every vertical morphism in C and is closed under horizontal composition. If double category C is strict, then, for every n , collection H_n^C is collection of morphisms of a category whose collection of objects is collection of vertical morphisms of C . In that case we call category H_n^C the n -th horizontal category associated to double category C . By the way sequence of vertical categories $\{V_n^C\}$ associated to C was constructed it is easily seen that for every n , inclusions $\text{Hom}V_n^C \subseteq H_{n+1}^C \subseteq \text{Hom}V_{n+1}^C$ hold. This implies that n -th vertical category V_n^C associated to double category C is a subcategory of $n+1$ -th vertical category V_{n+1}^C associated to C for every n . Moreover, in the case in which double category C is strict, n -th horizontal category H_n^C associated to C is a subcategory of $n+1$ -th horizontal category H_{n+1}^C associated to C for every n . The following lemma says that sequence of vertical categories of a globularly generated double category forms a filtration of its category of morphisms.

Lemma 3.2. *Let C be a globularly generated double category. Morphism category C_1 of C is equal to the limit $\varinjlim V_n^C$ in category adjacent to **Cat** of sequence $\{V_n^C\}$ of vertical categories associated to C .*

Proof. Let C be a globularly generated double category. We wish to prove that morphism category C_1 of C is equal to limit $\varinjlim V_n^C$ of sequence of vertical categories associated to C .

By the way sequence of vertical categories associated to C was constructed, it is easily seen that the union $\bigcup_{n=1}^{\infty} \text{Hom} V_n^C$ of sequence of its collections of morphisms is closed under the operations of taking vertical and horizontal compositions in C , and that it contains collection of horizontal identities of vertical morphisms of C . It follows, from this and from the requirement that components of identity transformations and associator of C are globular, that pair formed by category of objects C_0 of C and limit $\varinjlim V_n^C$ of sequence of vertical categories associated to C is a sub-double category of C . Collection of objects of n -th vertical category V_n^C associated to C is equal to collection of horizontal morphisms of C for every n . Thus collection of objects of category $\varinjlim V_n^C$ is equal to collection of horizontal morphisms of C . Pair formed by category of objects C_0 of C and limit $\varinjlim V_n^C$ of sequence of vertical categories associated to C is thus a complete sub-double category of C . Moreover, collection of morphisms $\bigcup_{n=1}^{\infty} \text{Hom} V_n^C$ of category $\varinjlim V_n^C$ contains collection of globular 2-morphisms of C . We conclude, from this, and from the fact that double category C is globularily generated, that sub-double category formed by C_0 and $\varinjlim V_n^C$ is equal to C and thus morphism category C_1 of C equals the limit $\varinjlim V_n^C$ of sequence of vertical categories associated to C . This concludes the proof. \blacksquare

Given a strict double category C pair τC formed by collection of vertical morphisms of C and collection of 2-morphisms of C is a category. Composition operation in pair τC is horizontal composition in C . We call category τC associated to a strict double category C the transversal category associated to C .

Corollary 3.3. *Let C be a globularily generated double category. If C is a strict double category, then transversal category τC associated to C is equal to the limit $\varinjlim H_n^C$, in category adjacent to \mathbf{Cat} , of sequence of horizontal categories associated to C .*

Proof. Let C be a strict globularily generated double category. We wish to prove, in this case, that transversal category τC associated to C is equal to the limit $\varinjlim H_n^C$ of sequence of horizontal categories associated to C .

By the way sequence of horizontal categories associated to globularily generated double category C was constructed, it is easily seen that collection of objects of n -th horizontal category H_n^C associated to C is equal to collection of vertical morphisms of C for every n . It follows, from this, that collection of objects of limit $\varinjlim H_n^C$ of sequence of horizontal categories associated to C is equal to collection of vertical morphisms of C and thus is

equal to collection of objects of transversal category τC of C . Collection of morphisms of limit $\varinjlim H_n^C$ is equal, to the union $\bigcup_{n=1}^{\infty} \text{Hom} H_n^C$ of collections of morphisms of horizontal categories associated to C . This union is equal to union $\bigcup_{n=1}^{\infty} \text{Hom} V_n^C$ of collections of morphisms of vertical categories associated to C , which by lemma 3.2 is equal to collection of 2-morphisms of C . This concludes the proof. \blacksquare

Definition 3.4. Let C be a globularly generated double category. Let Φ be a 2-morphism in C . We call the minimal integer n such that Φ is a morphism of n -th vertical category V_n^C associated to C the vertical length of Φ .

We now apply the concept of vertical length to the proof of results concerning the structure of globularly generated double categories. We first establish some notational conventions.

Assuming a double category C is strict, horizontal composition $\Phi_k * \dots * \Phi_1$ of any composable sequence Φ_1, \dots, Φ_k of 2-morphisms in C , is unambiguously defined. This is not the case in general. If double category C is not assumed to be strict, then horizontal compositions of a composable sequence Φ_1, \dots, Φ_k of 2-morphisms in C , following different parentheses patterns, might yield different 2-morphisms. If 2-morphism Φ in double category C can be obtained as horizontal composition, following a certain parentheses pattern, of composable sequence of 2-morphisms Φ_1, \dots, Φ_k , we will write $\Phi \equiv \Phi_k * \dots * \Phi_1$. Given 2-morphisms Φ and Ψ in a double category C , we say that Φ and Ψ are globularly equivalent if there exist globular 2-isomorphisms Θ_1, Θ_2 in C such that equation $\Phi = \Theta_1 \Psi \Theta_2^{-1}$ holds. From the fact that associators in double categories satisfy McLane's pentagon axiom and from the easy observation that collection of globular 2-morphisms of any double category is closed under the operations of taking vertical and horizontal composition, it follows that if two 2-morphisms Φ and Ψ satisfy equation $\Phi, \Psi \equiv \Phi_k * \dots * \Phi_1$ for a composable sequence of 2-morphisms Φ_1, \dots, Φ_k in C , then Φ and Ψ are globularly equivalent. Finally, we will say that a 2-morphism Φ in a double category C is a horizontal endomorphism, if source and target $s\Phi, t\Phi$ of Φ , are equal. Horizontal identities are examples of horizontal endomorphisms. The next proposition says that a 2-morphism in a globular double category is either globular or a horizontal endomorphisms.

Proposition 3.5. *Let C be a globularly generated double category. Let Φ be a 2-morphism in C . If Φ is not globular then Φ is a horizontal endomorphism.*

Proof. Let C be a globularly generated double category. Let Φ be a non-

globular 2-morphism in C . We wish to prove that Φ is a horizontal endomorphism.

We proceed by induction on the vertical length of Φ . Suppose first that Φ is an element of H_1^C . In that case, by the assumption that Φ is non-globular, Φ must be the horizontal identity of a vertical morphism in C and thus must be a horizontal endomorphism. Suppose now that Φ is a general element of first vertical category V_1^C associated to C . Write Φ as a vertical composition $\Phi = \Phi_k \circ \dots \circ \Phi_1$ where Φ_i is an element of H_1^C for every k . Moreover, assume that the length k of this decomposition is minimal. We prove by induction on k that Φ must be a horizontal endomorphism. Suppose first that $k = 1$. In that case Φ is an element of H_1^C and thus a horizontal endomorphism. Suppose now that k is strictly greater than 1 and that the result is true for every 2-morphism in first vertical category V_1^C associated to C that can be written as a vertical composition of strictly less than k 2-morphisms in H_1^C . Write Ψ for composition $\Phi_k \circ \dots \circ \Phi_2$. In this case equation $\Phi = \Psi \circ \Phi_1$ holds. Now, since collection of globular 2-morphisms of C is closed under the operation of taking vertical composition, one of Ψ and Φ_1 is not globular. If both Ψ and Φ_1 are not globular, then by induction hypothesis both Φ and Ψ are horizontal endomorphisms and thus their vertical composition Φ is a horizontal endomorphism. Suppose now that Ψ is globular. In that case Φ_1 is a horizontal endomorphism. Now, from the fact that source and target of Ψ are in this case vertical identities and from the fact that source and target are functorial, equations $s\Phi = s\Phi_1$ and $t\Phi = t\Phi_1$ follow and thus Φ is a horizontal endomorphism. The case in which Φ_1 is globular is handled analogously. This concludes the base of the induction.

Let n be strictly greater than 1. Assume now that every non-globular 2-morphism in C of vertical length strictly less than n is a horizontal endomorphism. Suppose first that Φ is an element of H_n^C . Let $\Phi_k * \dots * \Phi_1$ represent a horizontal composition in C such that Φ_i is an element of V_{n-1}^C for each k and such that $\Phi \equiv \Phi_k * \dots * \Phi_1$. Suppose the length k of this decomposition is minimal. We proceed by induction over k . If $k = 1$ then Φ is an element of V_{n-1}^C and is thus a horizontal endomorphism by induction hypothesis. Suppose now that k is strictly greater than 1 and that the result is true for every non-globular 2-morphism in H_n^C that can be written as a horizontal composition of strictly less than k 2-morphisms in V_{n-1}^C . Choose Ψ such that $\Psi \equiv \Phi_k * \dots * \Phi_2$. In this case Φ and $\Psi * \Phi_1$ are globularly equivalent and thus have the same source and target. Now, if both Ψ and Φ_1 are globular, then their horizontal composition, and every 2-morphism globularly equivalent to it, is globular. We thus assume that one of Ψ and Φ_1 is non-globular. If Ψ is globular, then equation $t\Phi_1 = s\Psi$ together with

induction hypothesis implies that Φ_1 is globular. An identical argument implies that if Φ_1 is globular Ψ is globular. We conclude that both Ψ and Φ_1 are non-globular and thus by induction hypothesis are horizontal endomorphisms. This and equation $t\Phi_1 = s\Psi$ implies that $\Psi * \Phi_1$ and thus Φ is a horizontal endomorphism. Assume now that Φ is a general element of V_n^C . Write Φ as a vertical composition $\Phi_k \circ \dots \circ \Phi_1$ where Φ is an element of H_n^C for every k . Moreover, assume again that the length k of this decomposition is minimal. An induction argument over k together with an argument analogous to that presented in the base of the induction proves that Φ is a horizontal endomorphism. This concludes the proof. \blacksquare

The following corollary follows immediately from the previous proposition.

Corollary 3.6. *Let C be a globularly generated double category. Let Φ and Ψ be 2-morphisms in C . Suppose Φ and Ψ are composable. In that case horizontal composition $\Psi * \Phi$ is a globular if and only if Φ and Ψ are both globular.*

We conclude this section with the following technical lemma.

Lemma 3.7. *Let C be a globularly generated double category. Let Φ be a 2-morphism in C . If vertical length of Φ is equal to 1 then Φ can be written as a vertical composition of the form*

$$\Psi_k \circ \Phi_k \circ \dots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

where Φ_i is a horizontal identity for every $1 \leq i \leq k$ and Ψ_i is globular for every $0 \leq i \leq k$.

Proof. Let C be a globularly generated double category. Let Φ be a 2-morphism in C . Suppose that the vertical length of Φ is equal to 1. We wish to prove, in this case, that Φ admits a decomposition as described in the statement of the lemma.

Suppose first that Φ is an element of H_1^C . In that case Φ is either globular or Φ is the horizontal identity of a vertical morphism in C . Suppose first that Φ is globular. In that case make $k = 0$ and $\Psi_0 = \Phi$. Suppose now that Φ is the horizontal identity of a vertical morphism α in C , with domain and codomain x and y respectively. In that case make $k = 1$, make Ψ_0 to equal to identity 2-morphism of horizontal identity of x , make Φ_1 to be equal to Φ and make Ψ_1 to be equal to identity 2-morphism of horizontal identity of y .

Suppose now that Φ is a general element of first vertical category V_1^C associated to C . Write Φ as the vertical composition $\Phi = \Theta_m \circ \dots \circ \Theta_1$,

where Θ_i is an element of H_1^C for every i . Choose this decomposition in such a way that its length m is minimal. We proceed by induction on m . In the case in which m is equal to 1 Φ is an element of H_1^C . Suppose now that m is strictly greater than 1 and that the result is true for every 2-morphism in V_1^C that can be written as a vertical composition of strictly less than m elements of H_1^C . Write Ψ for vertical composition $\Theta_{m-1} \circ \dots \circ \Theta_1$. In that case Ψ admits a decomposition as

$$\Psi = \Psi_k \circ \Phi_k \circ \dots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

for some k , where Φ_i is a horizontal identity for every $1 \leq i \leq k$ and Ψ_i is globular for every $0 \leq i \leq k$. Since Θ_m is an element of H_1^C then it is either globular or it is the horizontal identity of a vertical morphism in C . Suppose first that Θ_m is globular. In that case write Ψ'_k for vertical composition $\Theta_m \circ \Psi_k$. In that case decomposition

$$\Phi = \Psi'_k \circ \Phi_k \circ \dots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

satisfies the conditions of the lemma. Suppose now that Θ_m is the vertical identity of a vertical morphism α , with domain and codomain x and y respectively. In that case write Φ_{k+1} for Θ_m and write Ψ_{k+1} for the identity 2-endomorphism of identity horizontal endomorphism of x . In that case decomposition

$$\Phi = \Psi_{k+1} \circ \Phi_{k+1} \circ \dots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

satisfies the conditions of the lemma. This concludes the proof. \blacksquare

4 Globularily generated piece

In this section we explain how to associate, to every double category, a globularily generated double category, its globularily generated piece. We furnish the the globularily generated piece construction with the structure of a 2-functor and we prove that globularily generated piece 2-functor is a strict reflector. We regard the globularily generated piece construction as a globularily generated analog of horizontalization functor [7]. Finally, we categorize the vertical filtration construction to a filtration of globularily generated piece 2-functor.

Given a double category C , we write γC for sub-double category of C generated by collection of globular 2-morphisms in C . We call double category γC

the globularily generated piece of double category C . Globularily generated piece γC of double category C is globularily generated. Moreover, globularily generated piece γC of double category C is equal to both the maximal globularily generated sub-double category of C and to the minimal complete sub-double category of C containing collection of globular 2-morphisms of C . This last condition is equivalent to the following universal property characterizing globularily generated piece γC of C up to double isomorphisms: Given a double functor $F : D \rightarrow C$ from a globularily generated double category D to C , there exists a unique double functor $\tilde{F} : D \rightarrow \gamma C$ from D to globularily generated piece γC of C such that equation $\epsilon \tilde{F} = F$ holds, where ϵ denotes the inclusion double functor of globularily generated piece γC in C .

Given a double category C we call 2-morphisms in C lying in globularily generated piece γC of C globularily generated. We show how to extend the definition of the globularily generated piece of a double category to a 2-functor. Given double functors $F, G : C \rightarrow D$ from double category C to a double category D we say that a double natural transformation $\eta : F \rightarrow G$, from F to G , is globularily generated, if every component of morphism part η_1 of η is globularily generated. Identity natural transformations are examples of globularily generated double natural transformations. Further, collection of globularily generated double natural transformations is closed under the operations of taking vertical and horizontal compositions. It follows, from this, that triple formed by collection of double categories, collection of double functors, and collection of globularily generated natural transformations forms a sub 2-category of 2-category \mathbf{dCat} . We denote this 2-category by \mathbf{dCat}^g . Further, we denote by \mathbf{gCat} full sub 2-category of \mathbf{dCat}^g generated by globularily generated double categories. 2-category \mathbf{gCat} thus has collection of globularily generated double categories, collection of double functors between globularily generated double categories, and collection of globularily generated double natural transformations as collections of 0-, 1-, and 2-cells respectively. Observe that collection of globular 2-morphisms of a double category is invariant under the application of double functors. It follows, from this, that function associating, to every double category C , globularily generated piece γC of C , extends to a 2-functor from 2-category \mathbf{dCat}^g to 2-category \mathbf{gCat} . We call this 2-functor the globularily generated piece 2-functor. We denote the globularily generated piece 2-functor by γ . We consider 2-functor γ as a categorification of the globularily generated piece construction. Given 2-categories B and B' , such that B is a full sub 2-category of B' , we will say that B is strictly 2-reflective in B' if inclusion 2-functor of B in B' admits an left adjoint 2-functor with counit and unit

being strict 2-natural transformations [5]. In that case we will say that any 2-functor, left adjoint to inclusion 2-functor of B in B' , is a 2-reflector of B' on B . The next proposition says that 2-category \mathbf{gCat} as a sub 2-category of 2-category \mathbf{dCat}^g is strictly 2-reflective and that globularly generated piece 2-functor γ serves as a 2-reflector

Proposition 4.1. *2-category \mathbf{gCat} is a strictly 2-reflective sub 2-category of 2-category \mathbf{dCat}^g with globularly generated piece 2-functor γ as 2-reflector.*

Proof. Let i denote inclusion 2-functor of 2-category \mathbf{gCat} in 2-category \mathbf{dCat}^g . We wish to provide pair (γ, i) formed by globularly generated piece 2-functor γ and inclusion 2-functor i with the structure of an adjoint pair. We associate, to pair (γ, i) a counit-unit pair (ϵ, η) .

Let C be a double category. Write ϵ_C for inclusion double functor of globularly generated piece γC associated to C in C . We write ϵ for collection of inclusions ϵ_C where C runs through collection of all double categories. We prove that thus defined ϵ is a 2-natural transformation (see [5]) from composition $i\gamma$ of globularly generated piece 2-functor γ and inclusion i to identity 2-endofunctor $id_{\mathbf{dCat}^g}$ of 2-category \mathbf{dCat}^g . Let C and D be double categories. Let $F : C \rightarrow D$ be a double functor from C to D . Since double functor γ is defined by restriction on 1-cells, the following square

$$\begin{array}{ccc} i\gamma C & \xrightarrow{i\gamma F} & i\gamma D \\ \epsilon_C \downarrow & & \downarrow \epsilon_D \\ C & \xrightarrow{F} & D \end{array}$$

commutes. Now, let $F, G : C \rightarrow D$ be double functors from double category C to double category D and let $\mu : F \rightarrow G$ be a globularly generated natural transformation from F to G . We wish to prove that equation

$$\epsilon_D \mu = \mu \epsilon_C$$

holds. The above equation is equivalent to the following pair of equations

$$\epsilon_{D_0} \mu_0 = \mu_0 \epsilon_{C_0} \text{ and } \epsilon_{D_1} \mu_1 = \mu_1 \epsilon_{C_1}$$

Observe that since globularly generated piece 2-functor acts as the identity 2-functor on object categories, object functors, and object natural transformations, first equation above is trivial. We thus need only to prove that

second equation holds. Let α be a horizontal morphism in C . In that case μ_α is a globularly generated 2-morphism and thus $\epsilon_D \mu_\alpha$ is equal to μ_α . Now, $\epsilon_C \alpha$ is equal to α and thus $\mu \epsilon_C \alpha$ is equal to μ_α . We conclude that both equations above hold and thus ϵ is a strict 2-natural transformation from composition $i\gamma$ of globularly generated piece 2-functor γ and inclusion i to identity 2-endofunctor $id_{\mathbf{dCat}^g}$ of 2-category \mathbf{dCat}^g .

Now, since globularly generated piece of a globularly generated double category is equal to original globularly generated category and globularly generated piece double functor γ acts by restriction on double functors and double natural transformations, composition γi of inclusion i of 2-category \mathbf{gCat} in 2-category \mathbf{dCat}^g and globularly generated piece 2-functor γ is equal to identity 2-endofunctor of 2-category \mathbf{gCat} . Denote by η identity double natural transformation of identity 2-endofunctor $id_{\mathbf{gCat}}$ of \mathbf{gCat} as a natural transformation from $id_{\mathbf{gCat}}$ to composition γi . Thus defined η is a strict natural transformation. Finally, observe that from the way η was defined pair of natural transformations (ϵ, η) clearly satisfy the counit-unit triangle equations and it is thus a counit-unit pair for pair (γ, i) . We conclude that 2-category \mathbf{gCat} is a reflective 2-subcategory of 2-category \mathbf{dCat}^g and that 2-functor γ acts as a reflector. ■

We interpret proposition 4.1 by considering globularly generated piece 2-functor as a globularly generated analog of horizontalization functor. We now categorify the construction of filtration of vertical categories of a globularly generated double category introduced in section 3. We begin with the following lemma.

Lemma 4.2. *Let C and D be globularly generated double categories. Let $F : C \rightarrow D$ be a double functor from C to D . Let n be a positive integer. The image of n -th vertical category V_n^C associated to C , under morphism functor F_1 of F , is contained in n -th vertical category V_n^D associated to D . Moreover, in the case in which C, D , and F are strict, the image of n -th horizontal category H_n^C associated to C , under morphism functor F_1 of F , is contained in n -th horizontal category H_n^D associated to D .*

Proof. Let C and D be globularly generated double categories. Let $F : C \rightarrow D$ be a double functor from C to D . Let n be a positive integer. We wish to prove that the image of n -th vertical category V_n^C associated to C , under morphism functor F_1 of F , is a subcategory of n -th vertical category V_n^D associated to D . Moreover, we wish to prove that if C, D , and F are all strict then the image of n -th horizontal category H_n^C associated to C , under

morphism functor F_1 of F , is a subcategory of n -th horizontal category H_n^D associated to D .

We proceed by unduction on n . Let Φ be a 2-morphism in first vertical category V_1^C associated to C . We wish to prove, in this case that $F_1\Phi$ is a morphism in first vertical category V_1^D associated to D . Suppose first that Φ is an element of H_1^C . In that case Φ is either globular or the horizontal identity of a vertical morphism in C . Suppose first that Φ is the horizontal identity of a vertical morphism α in C . In that case the image $F_1\Phi$ of Φ under functor F_1 is globularly conjugate to horizontal identity of the image $F_0\alpha$ of α under functor F_0 and is thus a morphism in category V_1^D . Observe that in the case in which double functor F is strict $F_1\Phi$ is precisely horizontal identity of vertical morphism $F_0\alpha$ and is thus an element of H_0^D . From this and from the fact that collection of globular 2-morphisms of a double category is invariant under the application of double functors it follows that the image of collection H_1^C , under morphism functor F_1 , is contained in collection of morphisms of first vertical category V_1^D of D . Moreover, in the case in which F is strict, the image of H_1^C under F_1 is contained in H_1^D . Suppose now that Φ is a general element of first vertical category V_1^C associated to C . Write Φ as a vertical composition

$$\Phi = \Phi_k \circ \dots \circ \Phi_1$$

where Φ_i is an element of H_1^C for every $1 \leq i \leq k$. In that case the image of Φ under morphism functor F_1 of F is equal to vertical composition

$$F_1\Phi_k \circ \dots \circ F_1\Phi_1$$

which is a morphism of first vertical category V_1^D associated to D . Thus the image of first vertical category V_1^C associated to C , under morphism functor F_1 of F is a subcategory of first vertical category V_1^D associated to D . Moreover, if we assume that C, D , and F are strict, H_1^C and H_1^D are categories, and the image of H_1^C under morphism functor F_1 of F is a subcategory of H_1^D .

Let n now be strictly greater than 1. Suppose that the conclusions of the proposition are true for every $m < n$. Let Φ now be a morphism in n -th vertical category V_n^C associated to C . We wish to prove in this case that the image $F_1\Phi$ of Φ under morphism functor F_1 of F is a morphism in n -th vertical category V_n^D associated to D . Suppose first that Φ is a morphism in H_n^C . Write Φ , up to globular equivalences, as a horizontal composition of the form

$$\Phi \equiv \Phi_k * \dots * \Phi_1$$

where Φ_i is an element of $n-1$ -th vertical category V_{n-1}^C associated to C for every $1 \leq i \leq k$. In that case the image $F_1\Phi$ under functor F_1 is globularly equivalent to any possible interpretation of horizontal composition

$$F_1\Phi_k * \dots * F_1\Phi_1$$

in D . By the induction hypothesis $F_1\Phi_i$ is a morphism of $n-1$ -th vertical category V_{n-1}^D associated to D for every $1 \leq i \leq k$ and thus any interpretation of horizontal composition above is a morphism of n -th horizontal category V_n^D associated to D . We conclude that the image $F_1\Phi$ of 2-morphism Φ under functor F_1 is a morphism in n -th vertical category V_n^D associated to D . Moreover, if F is strict then image $F_1\Phi$ of Φ under functor F_1 is an element of H_n^D . Suppose now that Φ is a general morphism of n -th vertical category V_n^C associated to C . Write Φ as a vertical composition of the form

$$\Phi = \Phi_k \circ \dots \circ \Phi_1$$

where Φ_i is an element of H_n^C for every $1 \leq i \leq k$. In that case the image $F_1\Phi$ of Φ under morphism functor F_1 of F is equal to vertical composition

$$F_1\Phi_k \circ \dots \circ F_1\Phi_1$$

in D and thus is an element of n -th vertical category V_n^D of D . We conclude that the image of n -th vertical category V_n^C associated to C , under morphism functor of double functor F , is a subcategory of n -th vertical category V_n^D associated to D and that if C, D , and F are strict then moreover the image of n -th horizontal category H_n^C associated to C , under morphism functor F_1 of F , is a subcategory of n -th horizontal category H_n^D associated to D . This concludes the proof. \blacksquare

Let C and D be globularly generated double categories. Let $F : C \rightarrow D$ be a double functor from C to D . Let n be a positive integer. We write V_n^F for restriction, to n -th vertical category V_n^C associated to C of morphism functor F_1 of F . Thus defined V_n^F is, by lemma 4.2, a functor from n -th vertical category V_n^C associated to C to n -th vertical category V_n^D associated to D . We call V_n^F the n -th vertical functor associated to F . Pair formed by function associating n -th vertical category V_n^C associated to C to every double category C and n -th vertical double functor V_n^F to every double functor

F forms a functor from category adjacent to 2-category **gCat** of globularly generated double categories, double functors, and globularly generated double natural transformations to category adjacent to 2-category **Cat** of categories, functors, and natural transformations. We call functor V_n the n -th vertical functor.

Denote now by π_0 and π_1 2-functors from 2-category **dCat** of double categories, double functors, and double natural transformations to 2-category **Cat** of categories, functors, and natural transformations such that for every double category C , $\pi_0 C$ and $\pi_1 C$ are equal to object category C_0 of C and morphism category C_1 of C respectively, such that for every double functor F , $\pi_0 F$ and $\pi_1 F$ are equal to object functor F_0 of F and morphism functor F_1 of F respectively, and finally, such that for every double natural transformation η , $\pi_0 \eta$ and $\pi_1 \eta$ are equal to object natural transformation η_0 associated to η and morphism natural transformation η_1 associated to η . We call π_0 and π_1 object and morphism projections of **dCat** respectively. We keep denoting by π_0 and π_1 restrictions of object and morphism projections of 2-category **dCat**, to sub 2-category **gCat** of globularly generated double categories, double functors, and globularly generated double natural transformations. We write γ_1 for composition $\pi_1 \gamma$ of globularly generated piece 2-functor γ and morphism projection π_1 . Given a double category C and positive integers m and n such that $n \geq m$, m -th vertical category $V_m^{\gamma C}$ associated to globularly generated piece γC of C is a subcategory of n -th vertical category $V_n^{\gamma C}$ associated to γC . We write $\eta_{m,n}^{\gamma C}$ for the inclusion functor of category $V_m^{\gamma C}$ in $V_n^{\gamma C}$. Observe that given a double functor $F : C \rightarrow D$ from double category C to a double category D , the fact that vertical functors $V_m^{\gamma F}$ and $V_n^{\gamma F}$ associated to double functor γF are restrictions of morphism functor γF_1 of γF , implies that square:

$$\begin{array}{ccc} V_m^{\gamma C} & \xrightarrow{V_m^{\gamma F}} & V_m^{\gamma D} \\ \eta_{m,n}^{\gamma C} \downarrow & & \downarrow \eta_{m,n}^{\gamma D} \\ V_n^{\gamma C} & \xrightarrow{V_n^{\gamma F}} & V_n^{\gamma D} \end{array}$$

commutes. That is, if we denote by $\eta_{m,n}$ collection of inclusions $\eta_{m,n}^{\gamma C}$ with C running through collection of double categories, then $\eta_{m,n}$ is a natural transformation from composition $V_m \gamma$ of globularly generated piece 2-functor γ and m -th vertical functor V_m to composition $V_n \gamma$ of globularly generated

piece 2-functor γ and n -th vertical functor V_n . Sequence formed by functors $V_n\gamma$ together with collection formed by natural transformations $\eta_{m,n}$ forms a diagram in category adjacent to 2-category **Cat** with base in category adjacent to 2-category **dCat**. The following proposition says that the limit of this diagram is functor γ_1 . Its proof follows directly from lemma 3.2.

Proposition 4.3. *Functor γ_1 defined above is equal to the limit $\varinjlim V_n\gamma$ of diagram formed by sequence of vertical functors $V_n\gamma$ and collection natural transformations $\eta_{m,n}$.*

Let now C and D be strict globularly generated double categories and let $F : C \rightarrow D$ be a strict double functor from C to D . If n is a positive integer, then from the assumption that C, D , and F are strict, and from lemma 4.3, it follows that pair formed by morphism function of object functor F_0 of F and morphism function of morphism functor F_1 of F restrict to a functor from n -th horizontal category H_n^C associated to C to n -th horizontal category H_n^D associated to D . We denote this functor by H_n^F and we call it the n -th horizontal functor associated to double functor F . Denote by **dCat** sub 2-category of **dCat** generated by collection of strict double categories and collection of strict double functors between them and denote by **gCat** sub 2-category of **dCat** generated by collection of strict globularly generated double categories. Given a positive integer n , pair of functions associating, for every strict globularly generated double category C n -th horizontal category H_n^C associated to C and to every strict double functor F n -th horizontal functor H_n^F associated to F is a functor from category adjacent to 2-category **gCat** of strict globularly generated double categories, strict double functors, and double natural transformations, to category adjacent to 2-category **Cat** of categories, functors, and natural transformations.

Given a strict double category C , we denoted, in section 3, by τC category whose collection of objects is collection of vertical morphisms of C and whose collection of morphisms is collection of 2-morphisms of C . We called category τC the transversal category associated to C . Given a strict double functor $F : C \rightarrow D$ from strict double category C to a strict double category D , we denote by τF functor from transversal category τC associated to C to transversal category τD associated to D such that object and morphism functions of τF are morphism function of object functor F_0 associated to F and morphism function of morphism functor F_1 associated to F respectively. We call τF transversal functor associated to strict double functor F . Pair of functions associating transversal category τC to a double category C and transversal functor τF to double functor F forms a functor from category adjacent to 2-category **dCat** of strict double categories, strict

double functors, and double natural transformations, to category adjacent to 2-category **Cat** of categories, functors, and natural transformations. We denote this functor by τ . We call τ the transversal category functor. We keep denoting by τ restriction of transversal functor to category adjacent to 2-category **gCat**. We will denote composition $\tau\gamma$ of adjacent functor of globularly generated piece double functor γ and transversal functor τ by γ^τ . Given a strict globularly generated double category C and positive integers m and n such that $m \leq n$, m -th horizontal category H_m^C associated to C is a subcategory of n -th horizontal category H_n^C associated to C . In this case denote by $\nu_{m,n}^C$ inculsion functor of H_m^C in H_n^C .

Given a strict double functor $F : C \rightarrow D$, from a strict double category C to a strict double category D , for every pair of positive integers m and n such that $m \leq n$, diagram

$$\begin{array}{ccc} H_m^{\gamma^C} & \xrightarrow{H_m^{\gamma^F}} & H_m^{\gamma^D} \\ \nu_{m,n}^{\gamma^C} \downarrow & & \downarrow \nu_{m,n}^{\gamma^D} \\ H_n^{\gamma^C} & \xrightarrow{H_n^{\gamma^F}} & H_n^{\gamma^D} \end{array}$$

commutes. That is, if in this case we denote by $\nu_{m,n}$ collection formed by inclusions $\nu_{m,n}^{\gamma^C}$ with C running through collection of strict double categories, then $\nu_{m,n}$ is a natural transformation from composition $H_m\gamma$ of functor adjacent to globularly generated piece 2-functor γ and m -th horizontal functor to composition $H_n\gamma$ of globularly generated piece γ and n -th globularly generated piece functor H_n . Sequence formed by functors H_n together with collection of natural transformations $\nu_{m,n}$ forms a diagram in category adjacent to **Cat**, with base in category adjacent to **dCat**. The following proposition says that the limit of this diagram is functor γ^τ . Its proof now follows directly from the second part of lemma 3.3.

Proposition 4.4. *Functor γ^τ defined above is the limit $\varinjlim H_n\gamma$ formed by sequence of horizontal functors $H_n\gamma$ and collection of natural transformations $\nu_{m,n}$.*

5 Examples

In this final section we present explicit computations of globularly generated piece of double categories introduced in section 2. We begin with the

computation of globularly generated piece $\gamma\mathbf{Alg}$ of double category \mathbf{Alg} of algebras, algebra morphisms, bimodules, and equivariant bimodule morphisms.

We will write horizontal equivariant endomorphism (f, Φ, f) in double category \mathbf{Alg} simply as (f, Φ) . If an equivariant morphism in \mathbf{Alg} is written in this way it will be assumed it is a horizontal endomorphism. Given a complex algebra A and a left-right A -bimodule M , we say that M is an A -cyclic bimodule if M is generated, as an A -bimodule, by a single element intertwining left and right actions of A on M . Equivalently, bimodule M is A -cyclic if there exists a bimodule epimorphism $A \rightarrow M$. Algebra A considered as a left-right bimodule over itself is an example of a cyclic bimodule. Given algebras A and B and $f : A \rightarrow B$ a unital algebra morphism, morphism f induces, on every left-right B -bimodule M , the structure of a left-right A -bimodule through equation $axa' = f(a)xf(a')$ for every $x \in M$ and $a, a' \in A$. We will call this left-right A -bimodule structure on left-right B -bimodule M the A -bimodule structure on M induced by morphism f . Given algebras A and B , a left-right A -bimodule M and a left-right B -bimodule N , we say that an equivariant morphism $(f, \varphi) : M \rightarrow N$ from M to N , is **2-subcyclic** if there exists a cyclic A -submodule L of N and a cyclic B -submodule K of N such that inclusions $\text{Im}\varphi \subseteq L \subseteq K$ hold. Equivariant morphisms between algebras are examples of 2-subcyclic equivariant morphisms. In order to explicitly compute globularly generated piece $\gamma\mathbf{Alg}$ of double category \mathbf{Alg} , by proposition 3.5 we need only to compute collection of non-globular, globularly generated 2-morphisms between horizontal endomorphisms in \mathbf{Alg} . We begin with the following lemma.

Lemma 5.1. *Let A and B be algebras. Let M and M' be left-right A -bimodules and let N and N' be left-right B -bimodules. Let $(f, \varphi) : M \rightarrow N$ be an equivariant morphism from M to N and let $(f, \varphi') : M' \rightarrow N'$ be equivariant morphism from M' to N' . If both (f, φ) and (f, φ') are 2-subcyclic then relative tensor product $(f, \varphi \otimes_f \varphi')$ is 2-subcyclic.*

Proof. Let A and B be algebras. Let M and M' be left-right A -bimodules and let N and N' be left-right B -bimodules. Let $(f, \varphi) : M \rightarrow N$ and $(f, \varphi') : M' \rightarrow N'$ be equivariant morphisms from M to N and from M' to N' respectively. Suppose both (f, φ) and (f, φ') are 2-subcyclic. We wish to prove in this case that relative tensor product $(f, \varphi \otimes_f \varphi')$ is 2-subcyclic.

Let L, L' and K, K' be bimodules such that L and L' are A -cyclic submodules of N and N' respectively and such that K and K' are B -cyclic

submodules of N and N' respectively. Moreover, let L, L' and K, K' satisfy inclusions $\text{Im}\varphi \subseteq L \subseteq K$ and $\text{Im}\varphi' \subseteq L' \subseteq K'$ respectively. Relative tensor product $L \otimes_A L'$ is an A -cyclic submodule of $N \otimes_A N'$, relative tensor product $K \otimes_B K'$ is a B -cyclic submodule of $N \otimes_B N'$, $L \otimes_A L'$ is contained in $K \otimes_B K'$, and finally $\text{Im}\varphi \otimes_f \varphi'$ is contained in $L \otimes_A L'$. This concludes the proof. \blacksquare

Proposition 5.2. *Let A and B be algebras. Let M be left-right A -bimodule and let N be a left-right B -bimodule. In that case collection of non-globular globularly generated equivariant morphisms from M to N is precisely collection of non-globular 2-subcyclic equivariant morphisms from M to N . Moreover, every globularly generated equivariant morphism from M to N has vertical length equal to 1.*

Proof. Let A and B be algebras. Let M be a left-right A -bimodule and let N be a left-right B -bimodule. We wish to prove that collection of non-globular globularly generated equivariant morphisms from M to N is precisely collection of non-globular 2-subcyclic equivariant morphisms from M to N . Moreover, we wish to prove that every globularly generated equivariant morphism from M to N has vertical length equal to 1.

We prove first that every non-globular 2-subcyclic equivariant morphism from M to N is globularly generated. Let $(f, \varphi) : M \rightarrow N$ be non-globular and 2-subcyclic. Let K be a B -cyclic submodule of N and let L be an A -cyclic submodule of K such that inclusions $\text{Im}\varphi \subseteq L \subseteq K$ hold. Let j denote the inclusion of K in N . Let $\tilde{\varphi}$ denote the codomain restriction of φ to K . Thus defined j is a globular and equivariant morphism $(f, \tilde{\varphi})$ makes the following triangle:

$$\begin{array}{ccc} M & \xrightarrow{(f, \varphi)} & N \\ (f, \tilde{\varphi}) \downarrow & & \nearrow (id_B, j) \\ K & & \end{array}$$

commute. Denote now by j' inclusion of L in K and denote by $\tilde{\varphi}$ codomain restriction of φ to L . Thus defined j' is globular, and equivariant morphism $(f, \tilde{\varphi})$ makes the following triangle:

$$\begin{array}{ccc}
M & \xrightarrow{(f, \bar{\varphi})} & K \\
(f, \bar{\varphi}) \downarrow & \nearrow (id_A, j') & \\
L & &
\end{array}$$

commute. The following square:

$$\begin{array}{ccc}
M & \xrightarrow{(f, \varphi)} & N \\
(f, \bar{\varphi}) \downarrow & & \uparrow (id_B, j) \\
L & \xrightarrow{(id_A, j')} & K
\end{array}$$

is thus commutative. Commutativity of this square is clearly equivalent to commutativity of square:

$$\begin{array}{ccc}
M & \xrightarrow{(f, \varphi)} & N \\
(id_A, \bar{\varphi}) \downarrow & & \uparrow (id_B, j) \\
L & \xrightarrow{(f, j')} & K
\end{array}$$

Left and right hand sides of this last square are Globular. Finally, triangle:

$$\begin{array}{ccc}
L & \xrightarrow{(f, j')} & K \\
(f, f) \downarrow & \nearrow (id_B, j') & \\
K & &
\end{array}$$

commutes, which proves that equivariant morphism (f, j') is a morphism in first vertical category $V_1^{\mathbf{Alg}}$ associated to \mathbf{Alg} . We conclude that 2-subcyclic equivariant morphism (f, φ) is globularly generated and has vertical length equal to 1.

We now prove that every non-globular globularly generated equivariant morphism from M to N is 2-subcyclic. Let $(f, \varphi) : M \rightarrow N$ be non-globular and globularly generated. Assume first that (f, φ) is an element of $H_1^{\mathbf{Alg}}$. From the assumption that (f, φ) is non-globular it follows that (f, φ) is the horizontal identity of an algebra morphism and thus is 2-subcyclic. Suppose now that (f, φ) is a general morphism in first vertical category $V_1^{\mathbf{Alg}}$ associated to \mathbf{Alg} . We wish to find, in this case, an A -cyclic submodule L of N and a B -cyclic submodule K of N such that inclusions $\text{Im} \varphi \subseteq L \subseteq K$ hold. Write (f, φ) as a vertical composition of the form:

$$(id_B, \psi_{k+1}) \circ (f_k, \phi_k) \circ \dots \circ (f_1, \phi_1) \circ (id_A, \psi_1)$$

as in lemma 3.7 where f_i is an algebra morphism for every $i \leq k$. Write (f, Φ) for composition $(f_k, \phi_k) \circ \dots \circ (f_1, \phi_1)$. Thus defined (f, Φ) is an equivariant morphism from left-right A -bimodule ${}_A A_A$ to left-right B -bimodule ${}_B B_B$. Now make K to be equal to image $\text{Im} \psi_1$ of ψ_1 and make L to be equal to image $\text{Im} \Phi \psi$ of composition $\Phi \psi$. Thus defined K and L satisfy the conditions required. We conclude that every equivariant morphism in first vertical category $V_1^{\mathbf{Alg}}$ associated to \mathbf{Alg} is 2-subcyclic. From this and from lemma 5.1 it follows that every globularly generated equivariant morphism between M and N is 2-subcyclic. The fact that every non-globular globularly generated 2-morphism in \mathbf{Alg} has vertical length equal to 1 follows from this and from the first part of the proof. This concludes the proof. \blacksquare

We consider, due to proposition 3.5 that proposition 5.2 provides an explicit description of globularly generated piece $\gamma \mathbf{Alg}$ of double category \mathbf{Alg} . A similar computation provides a complete description of globularly generated piece $\gamma \mathbf{vN}^f$ of double category \mathbf{vN}^f of von Neumann algebras with finite dimensional center, finite algebra morphisms, bimodules, and equivariant bimodule morphisms. We now compute, using the same procedure used to compute globularly generated piece $\gamma \mathbf{Alg}$ of double category \mathbf{Alg} , globularly generated piece $\gamma \mathbf{Cob}(n)$ of double category $\mathbf{Cob}(n)$ of n -dimensional manifolds, diffeomorphisms, cobordisms, and equivariant diffeomorphisms, for every positive integer n . We make the same considerations regarding horizontal equivariant endomorphisms in $\mathbf{Cob}(n)$ as we did with horizontal equivariant endomorphisms in \mathbf{Alg} . We will say that cobordisms M and N from a closed manifold X to itself are globularly diffeomorphic if M and N are diffeomorphic relative to X . We begin with the following lemma.

Lemma 5.3. *Let n be a positive integer. Let X and Y be closed n -dimensional manifolds. Let M be a cobordism from X to X and let N be a cobordism from Y to Y . If there exist non-globular globularly generated diffeomorphisms from M to N then M and N are globularly diffeomorphic to identity cobordisms i_X and i_Y respectively.*

Proof. Let n be a positive integer. Let X and Y be closed n -dimensional manifolds. Let M be a cobordism from X to X and let N be a cobordism from Y to Y . Suppose there exists a non-globular globularly generated diffeomorphism from M to N . In that case we wish to prove that M and N are globularly diffeomorphic to identity cobordisms i_X and i_Y respectively.

Let $(f, \Phi) : M \rightarrow N$ be a non-globular globularly generated diffeomorphism from M to N . We proceed by induction on the vertical length of (f, Φ) to prove that the existence of (f, Φ) implies that M and N are globularly diffeomorphic to horizontal identities i_X and i_Y respectively. Suppose first that vertical length of (f, Φ) is equal to 1. By lemma 3.7 there exists a decomposition of (f, Φ) as a vertical composition of the form

$$(id_{X_k}, \Psi_k) \circ (f_k, f_k \times id_{[0,1]}) \circ \dots \circ (id_{X_1}, \Psi_1) \circ (f_1, f_1 \times id_{[0,1]}) \circ (id_{X_0}, \Psi_0)$$

where X_0, \dots, X_k are n -dimensional manifolds, X_0 and X_k are equal to X and Y respectively, $f_i : X_i \rightarrow X_{i+1}$ is a diffeomorphism from X_i to X_{i+1} for all $i \leq k-1$, and where Ψ_i is a globular diffeomorphism from X_i to X_i for all i . Since we assume that (f, Φ) is not globular then the length k of this decomposition is greater than or equal to 1. Domain of horizontal identity of Φ_1 is equal to horizontal identity i_X of manifold X and codomain of horizontal identity Φ_k is equal to horizontal identity i_Y of manifold Y . Thus Ψ_0 and Ψ_k define globular diffeomorphisms between M and N and horizontal identities i_X and i_Y respectively.

Let m be a positive integer strictly greater than 1. Assume now that the result is true for every pair of cobordisms admitting a non-globular globularly generated diffeomorphism of vertical length strictly less than m . Assume first that non-globular globularly generated diffeomorphism (f, Φ) is an element of $H_m^{\mathbf{Cob}(n)}$. Write, in this case (f, Φ) as a horizontal composition

$$(f, \Phi) \equiv (f, \Phi_k) * \dots * (f, \Phi_1)$$

where (f, Φ_i) is a morphism in $m-1$ -th vertical category $V_{m-1}^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$ for every $i \leq k$. Moreover, assume that length k of this

decomposition is minimal. We proceed by induction on k to prove that in this case the existence of (f, Φ) implies that M and N satisfy the conditions of the lemma. If $k = 1$ then (f, Φ) is an element of $m - 1$ -th vertical category $V_{m-1}^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$ and by induction hypothesis its existence implies that M and N satisfy the conditions of the lemma. Suppose now that k is strictly greater than 1. Write (f, Ψ) for any representative of $(f, \Phi_k) * \dots * (f, \Phi_2)$. In that case horizontal composition $(f, \Psi) * (f, \Phi_1)$ is equivalent to (f, Φ) . From the assumption that (f, Φ) is not globular and from corollary 3.6 it follows that non of its conjugate morphisms is globular. Thus horizontal composition $(f, \Psi) * (f, \Phi_1)$ is not globular and thus, again by corollary 3.6 neither of (f, Ψ) or (f, Φ_1) is globular. Both (f, Ψ) and (f, Φ_1) are globularly equivalent to the horizontal composition of strictly less than k morphisms in $m - 1$ -th vertical category $V_{m-1}^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. Let M_1 and N_1 be domain and codomain of (f, Φ_1) and let M_2 and N_2 be domain and codomain of (f, Ψ_1) . By induction hypothesis M_1 and M_2 are both globularly diffeomorphic to horizontal identity i_X of X and both N_1 and N_2 are globularly diffeomorphic to horizontal identity i_Y of Y . It follows that $M_2 * M_1$ is globularly diffeomorphic to horizontal identity i_X of X and that $N_2 * N_1$ is globularly diffeomorphic to horizontal identity i_Y of Y . Finally, by the exchange property in $\mathbf{Cob}(n)$ we conclude that M and N are globularly diffeomorphic to horizontal identities i_X and i_Y of X and Y respectively.

Suppose now that (f, Φ) is a general element of m -th vertical category $V_m^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. In that case write (f, Φ) as a vertical composition

$$(f, \Phi) = (f_k, \Phi_k) \circ \dots \circ (f_1, \Phi_1)$$

where (f_i, Φ_i) is an element of $H_m^{\mathbf{Cob}(n)}$ for every i . Moreover, assume that length k of this decomposition is minimal. We again proceed by induction on k . If $k = 1$ then (f, Φ) is an element of $H_m^{\mathbf{Cob}(n)}$. Suppose now that k is strictly greater than 1 and that the existence of a non-globular globularly generated diffeomorphism in m -th vertical category $V_m^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$, between manifolds X and Y , that can be written as a vertical composition of strictly less than k diffeomorphisms in $H_m^{\mathbf{Cob}(n)}$ implies the conclusion of the lemma for X and Y . Write (g, Ψ) for composition $(f_k, \Phi_k) \circ \dots \circ (f_2, \Phi_2)$. In that case (f, Φ) is equal to vertical composition $(g, \Psi) \circ (f_1, \Phi_1)$. Moreover, from the assumption that (f, Φ) is not globular it follows that one of (g, Ψ) or (f_1, Φ_1) is non-globular. Assume first that

(g, Ψ) is globular. In that case source and target of (f_1, Φ_1) are both equal to f . By induction hypothesis domain and codomain of (f, Φ_1) are globularly diffeomorphic to horizontal identity i_X of X and horizontal identity i_Y of Y respectively. Domain of (f, Φ) is equal to codomain of (f, Φ_1) and (g, Ψ) defines a globular diffeomorphism between codomain of (f, Φ) and codomain of (f_1, Φ_1) . We conclude that in this case, the existence of non-globular globularly generated diffeomorphism (f, Φ) implies the existence of a globular diffeomorphism between M and horizontal identity i_X of X and between N and horizontal identity i_Y of Y . The case in which it is assumed that (f_1, Φ_1) is globular is handled analogously. Suppose now that neither (g, Ψ) nor (f_1, Φ_1) are globular. In that case, induction hypothesis implies that there exists a globular diffeomorphism between M , which is the domain of (f_1, Φ_1) , and horizontal identity i_X of X and that there exists a globular diffeomorphism between N , which is the codomain of (g, Ψ) , and horizontal identity i_Y of Y . This concludes the proof. \blacksquare

As a consequence of lemma 3.5 and proposition 5.3, in order to compute globularly generated piece $\gamma\mathbf{Cob}(n)$ of double category $\mathbf{Cob}(n)$ it is enough to compute collection of non globular globularly generated diffeomorphisms between horizontal endomorphisms globularly diffeomorphic to horizontal identities of closed n -dimensional manifolds. This is achieved in the following proposition.

Proposition 5.4. *Let n be a positive integer. Let X and Y be closed n -dimensional manifolds. Let M be a cobordism from X to X and let N be a cobordism from Y to Y . Suppose that N is globularly diffeomorphic to identity cobordism i_Y associated to Y and that M is globularly diffeomorphic to identity cobordism i_X associated to X . In that case every horizontal endomorphism from M to N , in double category $\mathbf{Cob}(n)$, is globularly generated and has vertical length equal to 1.*

Proof. Let n be a positive integer. Let X and Y be closed n -dimensional manifolds. Let M be a cobordism from X to X , globularly diffeomorphic to horizontal identity i_X associated to manifold X and let N be a cobordism from Y to Y , globularly diffeomorphic to horizontal identity i_Y associated to Y . We wish to prove, in this case, that every 2-morphism, in double category $\mathbf{Cob}(n)$, from M to N , is globularly generated and has vertical length equal to 1.

We first prove the proposition for the case in which M and N are equal to horizontal identity cobordisms i_X and i_Y respectively. Let $(f, \Phi) : i_X \rightarrow i_Y$ be a 2-morphism, in $\mathbf{Cob}(n)$, from i_X to i_Y . In that case equivariant

morphism $(id_X, (f^{-1} \times id_{[0,1]})\Phi)$ is a globular endomorphism of horizontal identity i_X of X making the following triangle

$$\begin{array}{ccc}
 i_X & \xrightarrow{(f, \Phi)} & i_Y \\
 (id_X, (f^{-1} \times id_{[0,1]})\Phi) \downarrow & \nearrow (f, f \times id_{[0,1]}) & \\
 i_X & &
 \end{array}$$

commute. Since $(id_X, (f^{-1} \times id_{[0,1]})\Phi)$ is globular and $(f, f \times id_{[0,1]})$ is horizontal identity i_f of diffeomorphism f of X , we conclude that (f, Φ) is globularly generated and that its vertical length is equal to 1.

Suppose now M is a general cobordism from X to X globularly diffeomorphic to horizontal identity i_X of X and that N is a general cobordism from Y to Y globularly diffeomorphic to horizontal identity i_Y of Y . Let $(f, \Phi) : M \rightarrow N$ be a general 2-morphism, in $\mathbf{Cob}(n)$, from M to N . Let $(id_X, \varphi) : M \rightarrow i_X$ be a globular diffeomorphism from M to horizontal identity i_X of X and let $(id_Y, \phi) : N \rightarrow i_Y$ be a globular diffeomorphism from N to horizontal identity i_Y of Y . In that case composition $(f, \Psi) = (id_Y, \phi)(f, \Phi)(id_X, \varphi^{-1})$ is a 2-morphism from horizontal identity i_X of X to horizontal identity i_Y of Y and is thus a morphism in first vertical category $V_1^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. We conclude that $(f, \Phi) = (id_Y, \phi^{-1})(f, \Psi)(id_X, \varphi)$ is also a morphism in first vertical category $V_1^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. This concludes the proof. \blacksquare

By lemma 3.5 proposition 5.4 provides an explicit description of globularly generated piece of double categories of the form $\mathbf{Cob}(n)$. Examples of globularly generated double categories having 2-morphisms of vertical length strictly greater than 1 will be studied in subsequent papers.

6 Bibliography

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